

Perturbative super-Yang-Mills from the topological $AdS_5 \times S^5$ sigma model

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ABSTRACT: A topological sigma model based on the pure spinor formalism was recently proposed for the small radius limit of the $AdS_5 \times S^5$ superstring. Physical states in this model can be constructed by connecting holes on the worldsheet with Wilson lines of the worldsheet gauge field. The contribution of these states to the topological amplitude is claimed to reproduce the usual Feynman diagram expansion of gauge-invariant super-Yang-Mills correlation functions.

KEYWORDS: AdS-CFT Correspondence, Topological Strings.

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1. Introduction

The pure spinor formalism can be used to covariantly describe the superstring in any consistent $d = 10$ supergravity background [1]. When the supergravity background is $AdS_5 \times S^5$, the resulting worldsheet action has manifest $PSU(2, 2|4)$ symmetry and is constructed from the Metsaev-Tseytlin left-invariant currents $g^{-1}dg$ where g takes values in the coset $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$ [2]. In the large radius limit where $r_{AdS} \rightarrow \infty$, this action can be covariantly quantized [3, 4] and one can compute $PSU(2, 2|4)$ -covariant correlation functions as an expansion in $\frac{1}{r_{AdS}}$ [5]. However, to compare with computations in perturbative super-Yang-Mills, one needs to be able to quantize the worldsheet action in the small radius limit where $r_{AdS} \rightarrow 0$.

Recently, a proposal was made for how to quantize in the small radius limit [6, 7]. After combining the 22 pure spinor ghosts λ^α and $\hat{\lambda}^{\hat{\alpha}}$ with the ten $AdS_5 \times S^5$ spacetime variables into a 32-component unconstrained bosonic spinor, the $AdS_5 \times S^5$ worldsheet action was expressed as an $N = (2, 2)$ worldsheet supersymmetric action based on the fermionic coset $\frac{PSU(2,2|4)}{SU(2,2) \times SU(4)}$. This coset contains 32 fermionic variables, and the 32-component unconstrained bosonic spinor is the worldsheet superpartner of these variables.

If the BRST charge is defined to be the scalar worldsheet supersymmetry generator, this worldsheet supersymmetric action is a topological A-model which can be quantized using standard topological methods. However, in the large radius limit, it is important to note that the BRST charge defined in the pure spinor formalism is *not* the scalar worldsheet supersymmetry generator. So in the large radius limit, the $AdS_5 \times S^5$ worldsheet action is

not a topological A-model, which is expected since one has a continuum of physical states in this supergravity limit.

Nevertheless, it was conjectured that in the small radius limit, the BRST charge can be defined to be the scalar worldsheet supersymmetry generator such that the worldsheet action for the $AdS_5 \times S^5$ superstring becomes a topological A-model when $r_{AdS} \rightarrow 0$. Preliminary evidence for this conjecture came from an analogy with the Gopakumar-Vafa duality relating $d = 3$ Chern-Simons theory and the resolved conifold [8]. This open-closed duality was proven in [9] using a topological A-model and has many similarities with super-Yang-Mills/ $AdS_5 \times S^5$ duality. More recently, additional evidence for the conjecture was provided by Bonelli and Safaai [10] who argued that topological amplitudes involving certain D-branes in the model compute correlation functions of circular super-Yang-Mills Wilson lines. These D-branes break $PSU(2, 2|4)$ to $Osp(2, 2|4)$ which are the symmetries preserved by the circular Wilson lines.

If the conjecture is correct that this topological A-model describes the small radius limit of $AdS_5 \times S^5$, it should be possible to compute correlation functions of arbitrary gauge-invariant super-Yang-Mills operators using topological string methods. In this paper, it will be argued that topological amplitudes in this model indeed can compute arbitrary gauge-invariant super-Yang-Mills correlation functions. The topological amplitudes reproduce the usual perturbative Feynman diagram method for computing these correlation functions by replacing the propagators and vertices of Feynman diagrams with a network of Wilson lines of a worldsheet gauge field which connect holes on the closed string worldsheet.

The first step in computing these topological amplitudes is to note that the BRST-invariant topological A-model of [6, 7] can be expressed as the gauge-fixed version of a \mathcal{G}/\mathcal{G} principal chiral model where $\mathcal{G} = PSU(2, 2|4)$.¹ This principal chiral model is defined by the worldsheet action

$$S = Tr \int d^2z \left[r_{AdS}^2 (g^{-1} \bar{\nabla} g)(g^{-1} \nabla g) + \frac{1}{e^2} F^2 \right] \tag{1.1}$$

where g takes values in $PSU(2, 2|4)$, the covariant derivative on g is gauged using a $PSU(2, 2|4)$ worldsheet gauge field (A, \bar{A}) whose field-strength is F , and the infrared limit $e \rightarrow \infty$ is taken at the end of the computation.

If r_{AdS} is large, one can freely set $\frac{1}{e^2} = 0$ and the model becomes trivial by gauging away g such that the action reduces to $S = Tr \int d^2z r_{AdS}^2 A \bar{A}$. However, when r_{AdS} is small, there can be non-trivial fluctuations of the gauge field that survive in the limit where $e \rightarrow \infty$. These fluctuations are of size $(e r_{AdS})^{-1}$ and can be described by closed string vertex operators on the worldsheet which are connected to each other by a network of Wilson lines.

For the configuration corresponding to M gauge-invariant super-Yang-Mills operators, one will have M vertex operators on the closed string worldsheet. And if the r^{th} gauge-invariant operator is $Tr(\Phi_1 \dots \Phi_{n_r})$ where $\Phi_1 \dots \Phi_{n_r}$ are linearized super-Yang-Mills fields, there will be n_r Wilson lines emerging from the r^{th} hole which join with the Wilson lines

¹Based on analysis using the RNS formalism, a similar topological description of the zero radius limit of the $AdS_5 \times S^5$ superstring was discussed by Polyakov at Strings 2002 [11].

emerging from the other holes. This network of Wilson lines will represent a Feynman diagram of perturbative super-Yang-Mills, and it will be required that Wilson lines do not cross on the worldsheet so that the Feynman diagram can be thickened as in the ‘t Hooft large N expansion. Furthermore, it will be claimed that the contribution of each network to the topological amplitude coincides with the corresponding Feynman rules including the factor of $(\lambda_{\text{YM}}^2)^{2g-2}(\lambda_{\text{tHooft}})^{\text{faces}} = (\lambda_{\text{string}})^{2g-2}(r_{\text{AdS}}^4)^{\text{faces}}$ which is predicted by the Maldacena conjecture [12].

Note that in the topological amplitude computation, there is no integration over the locations of the closed string vertex operators. Unlike the proposal of [13] where the Schwinger parameters come from integration over worldsheet moduli, integrals over loop momenta in this description come from summing over the components in the singleton representation of $\text{PSU}(2, 2|4)$ which describe the propagating states in the Feynman diagram. This is similar to computations in twistor-string theory [14] [15] where tree-level super-Yang-Mills amplitudes are reproduced without any integration over worldsheet moduli.

An interesting question is how these topological amplitude computations are related to the usual prescription for closed superstring scattering amplitudes in the pure spinor formalism. Since three-point amplitudes of half-BPS states should be independent of r_{AdS} , the computation of these three-point amplitudes should be similar in the topological string prescription and in the pure spinor formalism.

In a flat background using the pure spinor formalism, integration over the left and right-moving worldsheet zero modes implies that non-vanishing correlation functions require 3 λ 's and 3 $\hat{\lambda}$'s as well as 5 θ 's and 5 $\hat{\theta}$'s in the combination [1]

$$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)(\hat{\lambda}\gamma^q\hat{\theta})(\hat{\lambda}\gamma^r\hat{\theta})(\hat{\lambda}\gamma^s\hat{\theta})(\hat{\theta}\gamma_{qrs}\hat{\theta}). \tag{1.2}$$

In an $AdS_5 \times S^5$ background using the pure spinor formalism, it will be argued that the analogous zero mode measure factor is simply

$$(\eta_{\alpha\hat{\beta}}\lambda^\alpha\hat{\lambda}^{\hat{\beta}})^3 \tag{1.3}$$

where $\eta_{\alpha\hat{\beta}} \equiv \gamma_{\alpha\hat{\beta}}^{01234}$. Moreover, for half-BPS states, the unintegrated closed string vertex operator is

$$V = (\eta_{\alpha\hat{\beta}}\lambda^\alpha\hat{\lambda}^{\hat{\beta}})f(x, \theta, \hat{\theta}) + \dots \tag{1.4}$$

where \dots is determined by BRST invariance. Since the three-point tree amplitude prescription using the pure spinor formalism is $\mathcal{A} = \langle V_1 V_2 V_3 \rangle$, one finds that after integrating over the pure spinor ghosts using the measure factor of (1.3), the pure spinor ghosts trivially decouple and the pure spinor computation reduces to the topological amplitude computation.

In section 2 of this paper, the topological A-model of [6, 7] is reviewed and is shown to be the gauge-fixed version of a \mathcal{G}/\mathcal{G} principal chiral model. In section 3, topological amplitudes in this model are shown to compute super-Yang-Mills Feynman diagrams in the ‘t Hooft large- N expansion. And in section 4, these topological amplitude computations are compared with closed superstring amplitude computations using the pure spinor formalism in an $AdS_5 \times S^5$ background.

2. Topological $AdS_5 \times S^5$ sigma model

2.1 Review of $\frac{PSU(2,2|4)}{SU(2,2) \times SU(4)}$ coset model

In [6] and [7], the pure spinor version of the superstring action in an $AdS_5 \times S^5$ background was mapped to an $N = (2, 2)$ worldsheet supersymmetric sigma model based on the coset $\frac{PSU(2,2|4)}{SU(2,2) \times SU(4)} = \frac{U(2,2|4)}{U(2,2) \times U(4)}$. Note that before introducing worldsheet gauge fields, the non-linear sigma model based on the coset $\frac{PSU(2,2|4)}{SU(2,2) \times SU(4)}$ is equivalent to the non-linear sigma model based on the coset $\frac{U(2,2|4)}{U(2,2) \times U(4)}$. It was more convenient in [7] to use the coset $\frac{U(2,2|4)}{U(2,2) \times U(4)}$ since the $U(1)$ gauge field of $U(4)$ was necessary for expressing the action as a gauged linear sigma model. In this paper, the gauged linear sigma model will not play any role and it will be necessary to use the coset $\frac{PSU(2,2|4)}{SU(2,2) \times SU(4)}$ so that the worldsheet gauge symmetries do not include the ‘‘bonus’’ $U(1)$ symmetry.

This non-linear sigma model was constructed from a fermionic coset G taking values in $\frac{PSU(2,2|4)}{SU(2,2) \times SU(4)}$ together with the bosonic ghosts $[Z_J^A, Y_A^J, \bar{Z}_A^J, \bar{Y}_J^A]$ where $A = 1$ to 4 is an $SU(2, 2)$ index and $J = 1$ to 4 is an $SU(4)$ index. The coset G can be parameterized as $G(\theta, \hat{\theta}) = e^{\theta^\alpha Q_\alpha + \hat{\theta}^{\hat{\alpha}} Q_{\hat{\alpha}}}$ where θ^α and $\hat{\theta}^{\hat{\alpha}}$ are 32 fermionic worldsheet scalars and, after performing an A -twist, (Z_J^A, \bar{Z}_A^J) are worldsheet scalars and Y_A^J and \bar{Y}_J^A carry conformal weight $(1, 0)$ and $(0, 1)$.

The map between these variables and the worldsheet variables of the pure spinor formalism can be found in [6, 7] and will not be necessary here. Up to a BRST-trivial term, this map takes the pure spinor version of the $AdS_5 \times S^5$ sigma model into the worldsheet action

$$\begin{aligned}
 S = r_{\text{AdS}}^2 \int d^2z [& (G^{-1} \partial G)_A^J (G^{-1} \bar{\partial} G)_J^A \\
 & - Y_A^J [\bar{\partial} Z_J^A + (G^{-1} \bar{\partial} G)_B^A Z_J^B - (G^{-1} \bar{\partial} G)_J^K Z_K^A] \\
 & + \bar{Y}_J^A [\partial \bar{Z}_A^J - (G^{-1} \bar{\partial} G)_A^B \bar{Z}_B^J + (G^{-1} \bar{\partial} G)_K^J \bar{Z}_A^K \\
 & \quad + Y_A^J Z_K^A \bar{Z}_B^K \bar{Y}_J^B - Z_J^A Y_B^J \bar{Y}_K^B \bar{Z}_A^K]. \quad (2.1)
 \end{aligned}$$

Although one can combine $(\theta^\alpha, \hat{\theta}^{\hat{\alpha}}, Z_J^A, \bar{Z}_A^J, Y_A^J, \bar{Y}_J^A)$ into $N = (2, 2)$ worldsheet superfields and write (2.1) in worldsheet superspace, it will be more convenient here to leave the worldsheet action in components.

It will be useful to note that by introducing the $SU(2, 2) \times SU(4)$ worldsheet gauge fields (A_B^A, \bar{A}_B^A) and (A_K^J, \bar{A}_K^J) , (2.1) can be written as

$$\begin{aligned}
 S = r_{\text{AdS}}^2 \int d^2z [& (G^{-1} \partial G)_A^J (G^{-1} \bar{\partial} G)_J^A + (G^{-1} \partial G - A)_B^A (G^{-1} \bar{\partial} G - \bar{A})_A^B \\
 & - (G^{-1} \partial G - A)_K^J (G^{-1} \bar{\partial} G - \bar{A})_J^K - Y_A^J (\bar{\nabla} Z)_J^A + \bar{Y}_J^A (\nabla \bar{Z})_A^J] \quad (2.2)
 \end{aligned}$$

where $(\bar{\nabla} Z)_J^A = \bar{\partial} Z_J^A + \bar{A}_B^A Z_J^B - \bar{A}_J^K Z_K^A$ and $(\nabla \bar{Z})_A^J = \partial \bar{Z}_A^J - A_A^B \bar{Z}_B^J + A_K^J \bar{Z}_A^K$. Although not manifest when written in components, (2.2) has $N = (2, 2)$ worldsheet supersymmetry and the $N = (2, 2)$ worldsheet superconformal generators are

$$Z_J^A (G^{-1} \partial G)_A^J, \quad Y_A^J (G^{-1} \partial G)_J^A, \quad \bar{Y}_J^A (G^{-1} \bar{\partial} G)_A^J, \quad \bar{Z}_A^J (G^{-1} \bar{\partial} G)_J^A. \quad (2.3)$$

So after performing an A -twist, the BRST operator in this topological A -model is identified with

$$Q = \int dz Z_J^A (G^{-1} \partial G)_A^J + \int d\bar{z} \bar{Z}_A^J (G^{-1} \bar{\partial} G)_J^A. \quad (2.4)$$

As explained in [7], the BRST operator of (2.4) for the topological A -model is not mapped into the BRST operator of the pure spinor formalism whose cohomology defines the physical spectrum at large r_{AdS} . Nevertheless, it was conjectured that at small r_{AdS} , the BRST operator of (2.4) can be used to define the physical states. This conjecture recently gained support from a paper showing that half-BPS super-Yang-Mills Wilson loops are described by D-branes in this topological A -model [10].

2.2 Principal chiral model

In this subsection, it will be shown that the action of (2.2) together with the BRST operator of (2.4) can be understood as a gauge-fixed version of the \mathcal{G}/\mathcal{G} principal chiral model where $\mathcal{G} = \text{PSU}(2, 2|4)$. So the pure spinor version of the $AdS_5 \times S^5$ sigma model can be mapped into a \mathcal{G}/\mathcal{G} principal chiral model. It will be also be shown that other gauge fixings of the \mathcal{G}/\mathcal{G} principal chiral model give rise to models based on the coset $\frac{\text{PSU}(2,2|4)}{\text{SU}(2|2) \times \text{SU}(2|2)}$ or $\frac{\text{PSU}(2,2|4)}{\text{SU}(1,1|2) \times \text{SU}(1,1|2)}$. Like the $\frac{\text{PSU}(2,2|4)}{\text{SU}(2,2) \times \text{SU}(4)}$ coset, these cosets are symmetric spaces and their actions are conformally invariant at the quantum level. However, unlike the $\frac{\text{PSU}(2,2|4)}{\text{SU}(2,2) \times \text{SU}(4)}$ coset which contains 32 fermions and no bosons, these cosets contain 16 bosons and 16 fermions.

The worldsheet action for the \mathcal{G}/\mathcal{G} principal chiral model is defined as

$$S = r_{\text{AdS}}^2 \int d^2 z (g^{-1} \partial g - A)_S^R (g^{-1} \bar{\partial} g - \bar{A})_R^S \quad (2.5)$$

where g takes values in $\text{PSU}(2, 2|4)$, $R = (A, J)$ is a $\text{PSU}(2, 2|4)$ index, and (A_S^R, \bar{A}_R^S) are worldsheet gauge fields taking values in the $\text{PSU}(2, 2|4)$ Lie algebra. Naively, this action is trivial since one can shift (A_S^R, \bar{A}_R^S) to eliminate g . However, as will be seen in the following section, non-trivial solutions can be obtained by introducing a kinetic term $\frac{1}{e^2} \int d^2 z F_S^R F_R^S$ for the worldsheet gauge field and taking the infrared limit $e \rightarrow \infty$ at the end of the computation.

The worldsheet action of (2.5) has a local $\text{PSU}(2, 2|4)$ gauge invariance under which $\delta g = g \Omega$ and $\delta A = d\Omega + [A, \Omega]$. To relate (2.5) to the action of (2.2), one should gauge-fix the $\text{SU}(2, 2) \times \text{SU}(4)$ subgroup of this invariance by choosing the gauge $g = G(\theta, \hat{\theta}) = e^{\theta^\alpha Q_\alpha + \hat{\theta}^{\hat{\alpha}} Q_{\hat{\alpha}}}$. Furthermore, one should gauge-fix the remaining 32 fermionic invariances by choosing the gauge

$$A_A^J = 0, \quad \bar{A}_J^A = 0 \quad (2.6)$$

for the fermionic worldsheet gauge fields.

The gauge choice $g = G(\theta, \hat{\theta})$ does not require Faddeev-Popov ghosts, however, the gauge choice of (2.6) requires the Faddeev-Popov ghosts (Z_J^A, \bar{Z}_A^J) and antighosts (Y_A^J, \bar{Y}_J^A) with the worldsheet action

$$S_{\text{ghost}} = \int d^2 z [-Y_A^J (\nabla \bar{Z})_J^A + \bar{Y}_J^A (\nabla \bar{Z})_A^J] \quad (2.7)$$

where $(\bar{\nabla}Z)_J^A$ and $(\nabla\bar{Z})_A^J$ are defined below (2.2). So after gauge-fixing, the worldsheet action is

$$\begin{aligned}
 S = r_{\text{AdS}}^2 \int d^2z & [(G^{-1}\partial G)_A^J (G^{-1}\bar{\partial}G)_J^A + (G^{-1}\partial G - A)_J^A (G^{-1}\bar{\partial}G - \bar{A})_A^J \\
 & + (G^{-1}\partial G - A)_B^A (G^{-1}\bar{\partial}G - \bar{A})_A^B - (G^{-1}\partial G - A)_K^J (G^{-1}\bar{\partial}G - \bar{A})_J^K \\
 & - Y_A^J (\bar{\nabla}Z)_J^A + \bar{Y}_J^A (\nabla\bar{Z})_A^J].
 \end{aligned} \tag{2.8}$$

Assuming that the kinetic term $\frac{1}{e^2} \int d^2z F_J^A F_A^J$ for the fermionic gauge fields A_J^A and \bar{A}_A^J can be ignored in the limit $e \rightarrow \infty$, one can integrate out these fermionic gauge fields to obtain the action of (2.2). Furthermore, the standard BRST quantization method implies that the BRST operator arising from the gauge-fixing of (2.6) is precisely (2.4).

So the $\frac{\text{PSU}(2,2|4)}{\text{SU}(2,2) \times \text{SU}(4)}$ worldsheet action and BRST operator can be understood as coming from the \mathcal{G}/\mathcal{G} principal chiral model in the gauge $A_J^A = \bar{A}_J^A = 0$. If one had instead chosen the gauge

$$A_{j'}^a = A_a^j = A_{\dot{a}}^a = A_{j'}^j = 0, \quad \bar{A}_a^{j'} = \bar{A}_{j'}^{\dot{a}} = \bar{A}_a^{\dot{a}} = \bar{A}_j^{j'} = 0, \tag{2.9}$$

where the $\text{SU}(2,2)$ and $\text{SU}(4)$ indices have been split into $\text{SU}(2) \times \text{SU}(2)$ and $\text{SU}(2) \times \text{SU}(2)$ indices as $A = (a, \dot{a})$ and $J = (j, j')$ for $a, \dot{a}, j, j' = 1$ to 2 , the resulting action and BRST operator would be constructed in a similar manner to (2.2) using the coset $\frac{\text{PSU}(2,2|4)}{\text{PS}[\text{U}(2|2) \times \text{U}(2|2)]}$. Similarly, if one had split the $\text{SU}(2,2)$ and $\text{SU}(4)$ indices into $\text{SU}(1,1) \times \text{SU}(1,1)$ and $\text{SU}(2) \times \text{SU}(2)$ indices, the resulting action and BRST operator would be constructed using the coset $\frac{\text{PSU}(2,2|4)}{\text{PS}[\text{U}(1,1|2) \times \text{U}(1,1|2)]}$.

So by starting with the \mathcal{G}/\mathcal{G} principal chiral model and choosing different gauge-fixings, one can relate topological A-models based on different symmetric coset spaces. Since the denominator of the coset determines the manifest symmetries, the worldsheet actions based on the $\frac{\text{PSU}(2,2|4)}{\text{PS}[\text{U}(2|2) \times \text{U}(2|2)]}$ and $\frac{\text{PSU}(2,2|4)}{\text{PS}[\text{U}(1,1|2) \times \text{U}(1,1|2)]}$ cosets may be useful for describing BPS states which preserve different symmetries than the half-BPS Wilson loops described in [10].

3. Feynman diagrams from topological model

3.1 Physical observables

As explained in the previous section, the topological A-model of [6, 7] can be understood as a gauge-fixed version of the \mathcal{G}/\mathcal{G} principal chiral model whose worldsheet action is

$$S = Tr \int d^2z [r_{\text{AdS}}^2 (g^{-1}\partial g - A)(g^{-1}\bar{\partial}g - \bar{A}) + \frac{1}{e^2} F^2] \tag{3.1}$$

where g takes values in $\text{PSU}(2,2|4)$, (A, \bar{A}) is a $\text{PSU}(2,2|4)$ worldsheet gauge field with field strength F , and one takes the infrared limit $e \rightarrow \infty$ at the end of the computation. Naively, this model has no physical states since one can use the local $\text{PSU}(2,2|4)$ symmetry to gauge $g = 1$ and, in the limit $e \rightarrow \infty$, the gauge field does not propagate.

Since the mass of the gauge field is $e r_{\text{AdS}}$, the fluctuations of the gauge field have size of order $(e r_{\text{AdS}})^{-1}$. If r_{AdS} is not small, the size of the fluctuations goes quickly to zero in

the infrared limit $e \rightarrow \infty$. However, if r_{AdS} is infinitesimal, these fluctuations may not be small and one can consider “holes” of size $(e r_{\text{AdS}})^{-1}$ in the worldsheet where the gauge field is nonzero.

Physical observables will be related to these fluctuations of the gauge field, and the locations of the “holes” will correspond to the locations of closed string vertex operators which carry global PSU(2, 2|4) indices. Since physical observables must be gauge invariant with respect to the local PSU(2, 2|4) symmetry, one needs to construct gauge-invariant operators out of g and A which describe these physical observables.

Under local PSU(2, 2|4) transformations parameterized by $\Omega_{J'}^{I'}$, the coset $g_{J'}^{I'}$ and the gauge field $A_{J'}^{I'}$ transform as

$$\delta g_{J'}^{I'} = g_{J'}^{J'} \Omega_{J'}^{I'}, \quad \delta A_{J'}^{I'} = d\Omega_{J'}^{I'} + A_{J'}^{K'} \Omega_{K'}^{I'} - \Omega_{J'}^{K'} A_{K'}^{I'}, \quad (3.2)$$

where I is a global PSU(2, 2|4) index and I' is a local PSU(2, 2|4) index. And under global PSU(2, 2|4) transformations parameterized by Σ_J^I ,

$$\delta g_I^{I'} = \Sigma_I^J g_J^{I'}, \quad \delta A_{J'}^{I'} = 0. \quad (3.3)$$

In general, the indices I and I' could label any representation of PSU(2, 2|4), however, throughout the rest of this paper the indices I and I' will always denote the “singleton” representation corresponding to the on-shell states of a super-Maxwell multiplet. The singleton representation is infinite-dimensional and it will be convenient to use the label $I = Z$ to denote the onshell scalar at zero momentum with +1 R-charge in the 56 direction of SO(6). All other states in the singleton representation can be obtained by repeatedly applying PSU(2, 2|4) transformations on this $I = Z$ state.

PSU(2, 2|4) gauge-invariant operators will be constructed with the help of the PSU(2, 2|4)-invariant tensors δ_{IJ} and ϵ_{IJK} where I, J, K indices always denote the singleton representation. If the index I denotes the super-Yang-Mills state ϕ_I , the tensors δ_{IJ} and ϵ_{IJK} are defined to be the free propagator and the bare three-vertex of super-Yang-Mills as

$$\delta_{IJ} = \langle \phi_I \phi_J \rangle, \quad \epsilon_{IJK} = \langle \phi_I \phi_J \phi_K \rangle \quad (3.4)$$

where the color indices of ϕ_I are ignored. An explicit construction of δ_{IJ} can be found in section (6.2) of [16] and section (3.1) of [17] where states in the singleton representation are mapped using a non-unitary transformation into states in position space. Once the singleton states are described in position space, one can use the standard definitions of the propagator δ_{IJ} and three-vertex ϵ_{IJK} . It will also be useful to define the tensor δ^{IJ} to be the inverse of δ_{IJ} which corresponds to the super-Yang-Mills kinetic operator. For example, if the indices I and J correspond to the scalars $Z_{[ij]}(x)$ and $Z_{[kl]}(y)$ where $i, j, k, l = 1$ to 4 are SU(4) indices and x^m and y^m label the point in $d = 4$,

$$\delta_{IJ} = \epsilon_{ijkl}(x - y)^{-2} \quad \text{and} \quad \delta^{IJ} = \epsilon^{ijkl} \partial_m \partial^m \delta^4(x - y). \quad (3.5)$$

And if I and J correspond to the chiral gluinos $\psi_i^\alpha(x)$ and $\psi_j^\beta(y)$ and K corresponds to the scalar $Z_{[kl]}(z)$,

$$\epsilon_{IJK} = \epsilon_{ijkl} \epsilon_{\dot{\alpha}\dot{\beta}} \int d^4 w F^{\alpha\dot{\alpha}}(x - w) F^{\beta\dot{\beta}}(y - w) G(z - w) \quad (3.6)$$

where $F^{\alpha\dot{\alpha}}(x-w) = \sigma_m^{\alpha\dot{\alpha}}(x-w)^m(x-w)^{-4}$ is the spinor propagator and $G(z-w) = (z-w)^{-2}$ is the scalar propagator.

Note that when expressed in terms of on-shell plane-wave states, these PSU(2, 2|4)-invariant tensors either vanish or become singular. For example, $\delta_{IJ} = p^{-2}\delta^4(p+q)$ and $\delta^{IJ} = p^2\delta^4(p+q)$ when expressed in terms of plane-wave scalar states with momenta p_m and q_m . To resolve these singularities, one needs to introduce a regulator which plays the role of the usual ($i\epsilon$) prescription in Feynman rules. Furthermore, one needs to convert sums over singleton indices into integrals over internal off-shell momenta. At the moment, it is unclear how to do this in a natural way.²

Up to overall normalization factors, δ_{IJ} and ϵ_{IJK} are the only independent PSU(2, 2|4) invariant tensors that can be constructed from the singleton representation. This follows from the fact that the $N = 4$ $d = 4$ super-Yang-Mills action is the unique PSU(2, 2|4)-invariant action, and the overall normalization of δ_{IJ} and ϵ_{IJK} can be absorbed by rescaling the super-Yang-Mills fields and the super-Yang-Mills coupling constant. Note that δ_{IJ} is invariant under the “bonus” U(1) symmetry which enlarges PSU(2, 2|4) to U(2, 2|4), however ϵ_{IJK} is not invariant under the “bonus” U(1) and is invariant only under PSU(2, 2|4).

At each “hole” in the worldsheet, the fluctuations of size $(e r_{\text{AdS}})^{-1}$ will be represented by a closed string vertex operator which carries global PSU(2, 2|4) indices and corresponds to a gauge-invariant super-Yang-Mills operator. At zero coupling constant, the gauge-invariant super-Yang-Mills operator can be described as a spin chain of L singleton representations which is invariant under cyclic permutations. Note that at zero coupling constant, PSU(2, 2|4) transformations act linearly on the super-Yang-Mills fields so that each singleton representation describes a single super-Yang-Mills field.

The closed string vertex operator at the r^{th} hole will have the form

$$V_r(z_r) = f_r^{I_1 \dots I_{L_r}} V_{I_1 \dots I_{L_r}}(z_r) \tag{3.7}$$

where $V_{I_1 \dots I_{L_r}}(z_r)$ is the vertex operator for the spin chain with L_r singleton representations and $f_r^{I_1 \dots I_{L_r}}$ are the “polarizations” of the fields in the r^{th} spin chain. Since $V_{I_1 \dots I_{L_r}}(z_r)$ carries L_r global PSU(2, 2|4) indices and is constructed from $g_I^{I'}$ and $A_{J'}^{I'}$, the only possibility is that $V_{I_1 \dots I_{L_r}}(z_r)$ is proportional to $g_{I_1}^{I'_1}(z_r) \dots g_{I_{L_r}}^{I'_{L_r}}(z_r)$.

In order to construct a physical observable which is invariant under local PSU(2, 2|4) transformations, each of the L_r primed indices $I'_1 \dots I'_{L_r}$ must be contracted with a path-ordered Wilson-line operator $P(\exp \int_{z_r} A)_{I'_i}^{J'_i}$ where the endpoint of the Wilson-line operator will be determined shortly. Furthermore, the L_r Wilson lines emerging from z_r will be prohibited from crossing and will be ordered clockwise such that they preserve the order of the indices on $V_{I_1 \dots I_{L_r}}$. This clockwise ordering implies that the vertex operator

$$V_{I_1 \dots I_{L_r}} = g_{I_1}^{I'_1}(z_r) (P e^{\int_{z_r} A})_{I'_1}^{J'_1} g_{I_2}^{I'_2}(z_r) (P e^{\int_{z_r} A})_{I'_2}^{J'_2} \dots g_{I_{L_r}}^{I'_{L_r}}(z_r) (P e^{\int_{z_r} A})_{I'_{L_r}}^{J'_{L_r}} \tag{3.8}$$

is invariant under cyclic permutations of the indices $I_1 \dots I_{L_r}$. The requirement that Wilson lines do not cross will be treated as an assumption, but the assumption might be justified

²I would like to thank Andrei Mikhailov and Warren Siegel for discussions on this point.

by the presence of singularities of crossing Wilson lines before taking the infrared limit $e \rightarrow \infty$.

Finally, to construct a gauge-invariant observable, one needs to contract the remaining J' index on each of the L_r Wilson lines which emerge from the r^{th} hole. These J' indices will be contracted either by joining the endpoints of two Wilson lines and contracting their J' and K' indices with the PSU(2, 2|4)-invariant tensor $\delta_{J'K'}$, or by joining the endpoints of three Wilson lines and contracting their J' , K' and L' indices using the PSU(2, 2|4)-invariant tensor $\epsilon_{J'K'L'}$. In the first case, the Wilson lines resemble a Feynman propagator connecting two super-Yang-Mills fields and, in the second case, the Wilson lines resemble a cubic vertex connecting three super-Yang-Mills fields. One can also construct gauge-invariant observables involving “internal” Wilson lines where both endpoints of the Wilson line are contracted with PSU(2, 2|4)-invariant tensors.³

3.2 Feynman diagrams

It will now be claimed that after taking the infrared limit $e \rightarrow \infty$, this network of vertex operators connected by Wilson lines reproduces the standard Feynman diagram computation in the 't Hooft large N expansion of perturbative super-Yang-Mills. Since the Wilson lines are prohibited from crossing on the worldsheet, the network of Wilson lines on a worldsheet of genus g corresponds to a thickened Feynman diagram of genus g . In the 't Hooft large N limit, the thickened Feynman diagram of genus g with F faces contributes a factor proportional to

$$N^{2-2g}(\lambda_{\text{tHooft}})^{F+2g-2} = (\lambda_{\text{YM}}^2)^{2g-2}(\lambda_{\text{YM}}^2 N)^F \tag{3.9}$$

where $\lambda_{\text{tHooft}} = \lambda_{\text{YM}}^2 N$. Since $\lambda_{\text{string}} = \lambda_{\text{YM}}^2$ and the genus g closed string amplitude is proportional to $(\lambda_{\text{string}})^{2g-2}$, the factor of (3.9) is reproduced if each face contributes a factor of λ_{tHooft} . Note that unlike the Chern-Simons/conifold duality where faces correspond to holes on the worldsheet, faces in this network are the regions bounded by Wilson lines and do not correspond to holes on the worldsheet.⁴

Extending the Maldacena conjecture to small r_{AdS} would imply that each face should contribute a factor of $\lambda_{\text{tHooft}} = r_{\text{AdS}}^4$. Although not rigorous, an argument which implies precisely such a contribution is as follows: After using the local PSU(2, 2|4) symmetry to gauge-fix $g_{J'}^I = \delta_{J'}^I$, the worldsheet action in the limit $e \rightarrow \infty$ is simply

$$S = r_{\text{AdS}}^2 \int d^2z A_{J'}^I \bar{A}_{I'}^{J'} \tag{3.10}$$

If one assumes that $A_{J'}^I$ can be discontinuous when crossing a Wilson line, the number of zero modes of $A_{J'}^I$ is equal to the number of faces in the network. Furthermore, the action of (3.10) implies that integration over each bosonic zero mode of A produces a factor of

³Andrei Mikhailov has pointed out that this network of Wilson lines resembles the network of transfer matrices considered in [18]. It would be very interesting to explore this relation, perhaps using the transfer matrices recently constructed in [19].

⁴I would like to thank Rajesh Gopakumar for stressing this point.

$(r_{\text{AdS}}^2)^{-1}$ and integration over each fermionic zero mode of A produces a factor of $(r_{\text{AdS}}^2)^{+1}$. Since the $\text{PSU}(2, 2|4)$ Lie algebra has 30 bosonic generators and 32 fermionic generators, the net contribution is a factor of r_{AdS}^4 for each face in the network. Note that for this argument to work, it is crucial that the gauge group is chosen to be $\text{PSU}(2, 2|4)$ as opposed to $\text{U}(2, 2|4)$, and this choice is also required by the fact that ϵ_{IJK} is not invariant under the bonus $\text{U}(1)$ symmetry.

Up to some subtleties mentioned at the end of this section, one can also argue that the network of Wilson lines connecting the vertex operators $V_{I_1 \dots I_{L_r}}(z_r)$ contributes to the topological amplitude using the same rules as the Feynman diagram connecting the gauge-invariant super-Yang-Mills operators described by $V_{I_1 \dots I_{L_r}}$. In the limit where $e \rightarrow \infty$, the equation of motion for the gauge field is $A = g^{-1}dg$. So after taking the limit $e \rightarrow \infty$, the path-ordered Wilson line operator connecting $g(y)$ and $g^{-1}(z)$ contributes

$$g_I^{I'}(y) P(e^{\int_y^z A})_{I'}^{J'} (g^{-1}(z))_{J'}^J = g_I^{I'}(y) P(e^{\int_y^z g^{-1}dg})_{I'}^{J'} (g^{-1}(z))_{J'}^J = \delta_I^J. \quad (3.11)$$

So the network of Wilson lines which connect the M vertex operators $V_r(z_r) = f_r^{I_1 \dots I_{L_r}} V_{I_1 \dots I_{L_r}}(z_r)$ contributes the topological amplitude

$$\mathcal{A} = \lambda_{\text{string}}^{2g-2} (r_{\text{AdS}}^4)^{\text{faces}} \left(\prod_{r=1}^M f_r^{I_1^{(r)} \dots I_{L_r}^{(r)}} \right) T_{I_1^{(1)} \dots I_{L_1}^{(1)} I_2^{(2)} \dots I_{L_2}^{(2)} \dots I_1^{(M)} \dots I_{L_M}^{(M)}} \quad (3.12)$$

where $T_{I_1^{(1)} \dots I_{L_M}^{(M)}}$ is a $\text{PSU}(2, 2|4)$ invariant tensor containing $\sum_{r=1}^M L_r$ indices which is constructed from the $\text{PSU}(2, 2|4)$ -invariant tensors δ_{IJ} , ϵ_{IJK} and δ^{IJ} . Since δ_{IJ} and ϵ_{IJK} correspond to the propagator and three-vertex of super-Yang-Mills, the tensor T computes the contribution of the super-Yang-Mills Feynman diagram which is described by the Wilson-line network. As expected from a topological amplitude computation, the amplitude of (3.12) is independent of the locations of the vertex operators and only depends on the topology of the Wilson-line network.

Using the above arguments, it seems reasonable to conjecture that the topological amplitude for the network of Wilson lines correctly reproduces the perturbative computation of gauge-invariant super-Yang-Mills correlation functions. However, there are several possible subtleties in proving this conjecture which deserve further study. Firstly, covariant Feynman diagram computations require gauge-fixing and ghosts, and the tensor T of (3.12) should somehow automatically include the ghost contributions. Secondly, loop computations require regularization, and one expects that a similar regularization for the tensor T is necessary when one has multiply contracted indices such as $\epsilon_{IJK} \delta^{KL} \epsilon_{LMN} \delta^{NI}$. Thirdly, the quartic vertex of super-Yang-Mills Feynman diagrams should somehow arise in T from a contact term when evaluating the contribution $\epsilon_{IJK} \delta^{KL} \epsilon_{LMN}$ that arises from the contraction of two cubic vertices. Note that after introducing auxiliary fields, the super-Yang-Mills action can be written as a cubic action. So it would not be surprising if the quartic vertex could be interpreted as a contact term of two cubic vertices coming from integrating out the auxiliary field.

4. Comparison with superstring amplitudes

4.1 $AdS_5 \times S^5$ measure factor

In the previous section, it was argued that perturbative super-Yang-Mills correlation functions can be computed as topological amplitudes using the small radius limit of the topological $AdS_5 \times S^5$ sigma model. These topological amplitude computations naively look very different from closed superstring amplitude computations using the pure spinor formalism. For example, in a flat background, unintegrated closed superstring vertex operators for supergravity states have the form $V = \lambda^\alpha \widehat{\lambda}^{\widehat{\alpha}} A_{\alpha\widehat{\alpha}}(x, \theta, \widehat{\theta})$ where λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$ are the left and right-moving pure spinor ghosts. And three-point amplitudes in a flat background are computed by $\mathcal{A} = \langle V_1 V_2 V_3 \rangle$ using the zero mode measure factor

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)(\widehat{\lambda}\gamma^q\widehat{\theta})(\widehat{\lambda}\gamma^r\widehat{\theta})(\widehat{\lambda}\gamma^s\widehat{\theta})(\widehat{\theta}\gamma_{qrs}\widehat{\theta}) \rangle = 1. \quad (4.1)$$

Since supergravity states in an $AdS_5 \times S^5$ background correspond to half-BPS super-Yang-Mills gauge-invariant operators, one expects that the three-point amplitude for these states should be independent of r_{AdS} . So it should be possible to relate the topological amplitude of this three-point half-BPS correlation function at small radius with the superstring amplitude computation at large radius. In this section, it will be shown how to relate these two computations.

The first step in relating the two computations is to determine the zero mode measure factor using the pure spinor formalism for the superstring in an $AdS_5 \times S^5$ background. This measure factor should be in the BRST cohomology at ghost-number $(3, 3)$ where the left and right-moving BRST operators are [4]

$$Q = \int dz \eta_{\alpha\widehat{\beta}} \lambda^\alpha (g^{-1}\partial g)^{\widehat{\beta}}, \quad \overline{Q} = \int d\overline{z} \eta_{\alpha\widehat{\beta}} \widehat{\lambda}^{\widehat{\beta}} (g^{-1}\overline{\partial} g)^\alpha, \quad (4.2)$$

$\eta_{\alpha\widehat{\beta}} = \gamma_{\alpha\widehat{\beta}}^{01234}$, and g takes values in the $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$ coset. Under the BRST transformations generated by (4.2),

$$\delta g = g(\lambda^\alpha T_\alpha + \widehat{\lambda}^{\widehat{\alpha}} T_{\widehat{\alpha}}), \quad \delta \lambda^\alpha = 0, \quad \delta \widehat{\lambda}^{\widehat{\alpha}} = 0, \quad (4.3)$$

where T_α and $T_{\widehat{\alpha}}$ are the 32 fermionic generators of $PSU(2, 2|4)$.

One clue in constructing the zero mode measure factor in an $AdS_5 \times S^5$ background is to note that for the Type IIA superstring in a flat background, the measure factor of (4.1) can be written as

$$\langle [(\lambda\gamma^m\theta)(\widehat{\lambda}\gamma_m\widehat{\theta})]^5 (\lambda^\alpha \widehat{\lambda}_\alpha)^{-2} \rangle = 1 \quad (4.4)$$

using the identities

$$(\lambda\gamma^{m_1}\theta)(\lambda\gamma^{m_2}\theta)(\lambda\gamma^{m_3}\theta)(\lambda\gamma^{m_4}\theta)(\lambda\gamma^{m_5}\theta) = (\lambda\gamma^{m_1\dots m_5}\lambda)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)(\theta\gamma_{npq}\theta) \quad (4.5)$$

and $(\lambda\gamma^{m_1\dots m_5}\lambda)(\widehat{\lambda}\gamma_{m_1\dots m_5}\widehat{\lambda}) = (\lambda^\alpha \widehat{\lambda}_\alpha)^2$ where overall proportionality factors are being ignored.

The operator $V_{\text{flat}} = (\lambda\gamma^m\theta)(\widehat{\lambda}\gamma_m\widehat{\theta})$ appearing in (4.4) is the vertex operator of the graviton trace at zero momentum, and is related to the worldsheet Lagrangian L_{flat} in a flat background by

$$Q\overline{Q}L_{\text{flat}} = \partial\overline{\partial}V_{\text{flat}}. \tag{4.6}$$

Using the worldsheet Lagrangian L_{AdS} for the pure spinor formalism in an $AdS_5 \times S^5$ background, one can similarly compute the vertex operator V_{AdS} for the AdS radius modulus at zero momentum and one finds that

$$Q\overline{Q}L_{\text{AdS}} = \partial\overline{\partial}V_{\text{AdS}} \tag{4.7}$$

where $V_{\text{AdS}} = \eta_{\alpha\widehat{\beta}}\lambda^\alpha\widehat{\lambda}^{\widehat{\beta}}$.

By analogy with the zero mode measure factor of (4.4), the natural guess for the zero mode measure factor in an $AdS_5 \times S^5$ background is therefore

$$\langle (\eta_{\alpha\widehat{\beta}}\lambda^\alpha\widehat{\lambda}^{\widehat{\beta}})^5 (\eta_{\gamma\widehat{\delta}}\lambda^\gamma\widehat{\lambda}^{\widehat{\delta}})^{-2} \rangle = \langle (\eta_{\alpha\widehat{\beta}}\lambda^\alpha\widehat{\lambda}^{\widehat{\beta}})^3 \rangle = 1. \tag{4.8}$$

So unlike in a flat background, the $AdS_5 \times S^5$ measure factor only involves the pure spinor ghosts and does not involve the matter fields. To verify that (4.8) is the correct measure factor, one can easily compute the tree amplitude of three radius moduli described by the vertex operator $V_{\text{AdS}} = \eta_{\alpha\widehat{\beta}}\lambda^\alpha\widehat{\lambda}^{\widehat{\beta}}$ and one finds that

$$\mathcal{A} = \langle V_{\text{AdS}}V_{\text{AdS}}V_{\text{AdS}} \rangle = 1. \tag{4.9}$$

Note that in a flat background, the analogous amplitude involving the zero momentum graviton trace vanishes since $(V_{\text{flat}})^3$ contains 3 θ 's and 3 $\widehat{\theta}$'s whereas the measure factor of (4.1) requires 5 θ 's and 5 $\widehat{\theta}$'s. This result is consistent with the fact that the $d = 10$ effective action vanishes in a flat background. But in an $AdS_5 \times S^5$ background, the effective action is a non-vanishing function of the AdS radius.

4.2 $AdS_5 \times S^5$ vertex operators

The next step in relating the computations of three-point half-BPS amplitudes is to construct the vertex operator for a general supergravity state in the pure spinor formalism. As explained in [3], one method for constructing the supergravity vertex operators uses a bispinor superfield $A_{\alpha\widehat{\beta}}(x, \theta, \widehat{\theta})$ satisfying the on-shell conditions

$$\gamma_{mnpqr}^{\alpha\gamma}\nabla_\gamma A_{\alpha\widehat{\beta}} = \gamma_{mnpqr}^{\widehat{\beta}\widehat{\gamma}}\nabla_{\widehat{\gamma}} A_{\alpha\widehat{\beta}} = 0 \tag{4.10}$$

where ∇_α and $\nabla_{\widehat{\alpha}}$ are the covariant fermionic derivatives in an $AdS_5 \times S^5$ background. As in a flat background, the unintegrated supergravity vertex operator in an $AdS_5 \times S^5$ background can be expressed in terms of $A_{\alpha\widehat{\beta}}$ as $V = \lambda^\alpha\widehat{\lambda}^{\widehat{\beta}}A_{\alpha\widehat{\beta}}(x, \theta, \widehat{\theta})$ and the on-shell conditions of (4.10) imply that $QV = \overline{Q}V = 0$.

From the analysis in the previous subsection, it is clear that the $\theta = \widehat{\theta} = 0$ component of $\eta^{\alpha\widehat{\beta}}A_{\alpha\widehat{\beta}}$ is the radius modulus, and other fields in the supergravity multiplet can be

obtained from this modulus by supersymmetry transformations. For example, the vertex operator for the scalar with J units of R-charge in the 56 direction is

$$V_J = (\eta_{\alpha\hat{\beta}} \lambda^\alpha \lambda^{\hat{\beta}}) a^{\pm J} e^{iJy_{56}} + \dots \quad (4.11)$$

where a is the x^5 direction in AdS_5 , y_{56} is the 56 direction in S^5 , the choice of \pm sign determines the AdS_5 boundary condition of the state, and \dots contains terms higher order in $(\theta, \hat{\theta})$ which are determined by BRST invariance.

If the plus sign is chosen in (4.11) so that V_J diverges as $a \rightarrow \infty$, the supergravity vertex operator corresponds to the $PSU(2, 2|4)$ representation with $|J|$ lowered indices. Using the notation where $I = Z$ corresponds to the zero-momentum scalar with $+1$ R-charge in the 56 direction and $I = \bar{Z}$ corresponds to the zero-momentum scalar with -1 R-charge in the 56 direction, $V_J = V_{Z\dots Z}$ when J is positive and $V_J = V_{\bar{Z}\dots\bar{Z}}$ when J is negative. On the other hand, if the minus sign is chosen in (4.11) so that V_J goes to zero as $a \rightarrow \infty$, the supergravity state corresponds to the $PSU(2, 2|4)$ representation with $|J|$ raised indices. Defining δ^{IJ} to be the same $PSU(2, 2|4)$ -invariant tensor defined earlier, $V_J = V^{\bar{Z}\dots\bar{Z}} = \delta^{\bar{Z}I_1} \dots \delta^{\bar{Z}I_J} V_{I_1\dots I_J}$ when J is positive and $V_J = V^{Z\dots Z} = \delta^{ZI_1} \dots \delta^{ZI_{|J|}} V_{I_1\dots I_{|J|}}$ when J is negative.

4.3 Three-point supergravity amplitude

Using the superstring vertex operators V_J of (4.11), it is easy to compare the three-point superstring tree amplitudes of these states with the topological amplitude computations. For the amplitude

$$\mathcal{A} = \langle V_{J_1}(z_1) V_{J_2}(z_2) V_{J_3}(z_3) \rangle, \quad (4.12)$$

the measure factor of (4.8) implies that $\mathcal{A} = 1$ if and only if $J_1 + J_2 + J_3 = 0$ and if the state with maximum $|J|$ charge has the opposite AdS_5 boundary condition from the other two states. These conditions guarantee that there are either an equal number of Z subscript and Z superscript indices on the vertex operators, or an equal number of \bar{Z} subscript and \bar{Z} superscript indices.

For example, suppose that J_1 is positive and J_2 and J_3 are negative such that $J_1 + J_2 + J_3 = 0$. If V_{J_1} diverges when $a \rightarrow \infty$, the amplitude $\langle V_{J_1}(z_1) V_{J_2}(z_2) V_{J_3}(z_3) \rangle = 1$ implies that

$$\langle V_{Z\dots Z}(z_1) \delta^{Z I_1} \dots \delta^{Z I_{|J_2|}} V_{I_1\dots I_{|J_2|}} \delta^{Z K_1} \dots \delta^{Z K_{|J_3|}} V_{K_1\dots K_{|J_3|}} \rangle = 1. \quad (4.13)$$

To show that this result agrees with the topological amplitude computation, note that for three-point amplitudes involving half-BPS states, only the propagator contributes to the Feynman diagram computation since the amplitude is independent of the super-Yang-Mills coupling constant. Since there are no contributions from cubic vertices, the topological amplitude computation involves a single Wilson-line network with J_1 propagators which contributes

$$\langle V_{Z\dots Z}(z_1) V_{I_1\dots I_{|J_2|}}(z_2) V_{K_1\dots K_{|J_3|}}(z_3) \rangle = \delta_{Z I_1} \dots \delta_{Z I_{|J_2|}} \delta_{Z K_1} \dots \delta_{Z K_{|J_3|}}. \quad (4.14)$$

So using $\delta_{ZI} \delta^{IZ} = 1$, one finds that (4.14) agrees with (4.13).

In comparing these topological amplitudes and pure spinor superstring amplitudes, it was important that the λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$ pure spinor ghosts decoupled in a trivial manner in the superstring computation. For amplitudes involving non-BPS states or more than three half-BPS states, the pure spinor ghosts probably play a more complicated role and it will be highly non-trivial to compare the two amplitude computations. This is not surprising since these amplitudes are expected to have non-trivial dependence on the AdS radius.

One situation which would be very interesting to study is the plane-wave limit in which the external vertex operators carry large R-charge. In this case, it might be possible to compare the topological and superstring computations for a more general class of scattering amplitudes. Perhaps in the limit of large R-charge, the discrete set of contributions to the topological amplitude combines into a continuous integral over worldsheet moduli in the superstring amplitude computation. Another speculation is that in the plane-wave limit, 8 bosonic and 8 fermionic components of the PSU(2, 2|4) gauge field might become dynamical and reproduce the light-cone degrees of freedom of the superstring.

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